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THE RIGIDITY OF SUSPENSIONS

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1. Introduction

We investigate the continuous rigidity of a suspension of a polygonal curve in three-space. The main result is that all such embedded suspensions are rigid. This is motivated by the old result of Cauchy (in 1813) [4] that all strictly convex polyhedral surfaces are rigid and by the conjecture that all embedded polyhedral surfaces are rigid.

We develop here some techniques which we feel are new to the subject of rigidity and apply them to obtain two results: We suppose that Σ is a suspension which flexes such that the distance between the suspension points changes, but of course all the edges have a constant length.

Theorem 1. *The winding number of the curve* (*equator*) *about the line through the suspension points is zero* (*when defined*).

For any polyhedral closed (orientable) surface in $\boldsymbol{R}^3$ it is possible to define the notion of a generalized volume, which is defined even if the surface is not embedded but only piecewise linearly mapped into $\boldsymbol{R}^3$. It agrees with the ordinary definition of the volume enclosed by the surface, when the surface is embedded.

Theorem 2. *For Σ as above, $V(\Sigma) = 0$, where $V(\Sigma)$ is the generalized volume.*

Theorem 2 implies that all embedded suspensions are rigid.

Recall from Gluck [6] that a polyhedron P regarded as a simplicial map, linear on each simplex, into $\boldsymbol{R}^3$ is *rigid* iff any homotopy P_t fixing the edge lengths (we call this a *flex*) is congruent in $\boldsymbol{R}^3$ to $P_0 = P$.

The proofs of the above involve first defining certain structural equations which describe the affine algebraic "variety" of the space of congruence classes of isometric maps of the polyhedral surface. This variety is described by certain extrinsic (variable) and intrinsic (constant) parameters, and in a sense Theorems 1 and 2 are formal consequences of the conditions of flexibility. The generalized volume and the winding number are particularly easy to analyse in the way we have chosen to set up the structural equations.

The analysis is based on the observation that the variety can be complexified in a natural way, and since we are interested when the polyhedron flexes we

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can interpret this to mean that the variety is at least (complex) one-dimensional. It turns out to be convenient to calculate what happens at infinity, and this is used to prove the theorems.

2. The structural equations

2.1. The vector equations. We state a relation which is helpful in describing the final equations.

Let A_1, A_2, A_3 be three vectors in $\boldsymbol{R}^3$, where A_1 and A_2 are not scalar multiples of $A_3 \neq 0$. Let $A_3^\perp$ denote the space perpendicular A_3. Let θ be the dihedral angle between the plane determined by A_1 and A_3, and the plane determined by A_2 and A_3. We regard θ as the angle between the orthogonal projection of A_1 and A_2 onto $A_3^\perp$. It is then an easy exercise to show

$$(1.1) \qquad e^{i\theta} = \frac{(A_1 \cdot A_2)(A_3 \cdot A_3) - (A_1 \cdot A_3)(A_2 \cdot A_3) + |A_3|\,[A_1, A_2, A_3]i}{|A_1 \times A_3|\,|A_2 \times A_3|}\,,$$

$$|A_j \times A_3|^2 = (A_j \cdot A_j)(A_3 \cdot A_3) - (A_j \cdot A_3)^2\,, \qquad j = 1, 2\,,$$

where $[X, Y, Z] = X \cdot Y \times Z$ is the scalar triple product. Let A denote the matrix with rows A_1, A_2, A_3. Then

$$(1.2) \qquad \begin{aligned}[A_1, A_2, A_3]^2 &= [\det(A_1, A_2, A_3)]^2 \\ &= (\det A)^2 = (\det A)(\det(A^t)) = \det((A_j \cdot A_k))\,,\end{aligned}$$

where the superscript t denotes the transpose of a matrix.

2.2. The Parameters describing the flex. We describe the suspension as follows: Let $v_1, v_2, \cdots, v_n$ be the vertices of the equator in order. Let the north and south poles N, S denote the suspension points. Thus $\langle v_j, v_{j+1}\rangle$, $\langle N, v_j\rangle$, $\langle S, v_j\rangle$ (j mod n) are the edges of the suspension Σ for $j = 1, \cdots, n$.

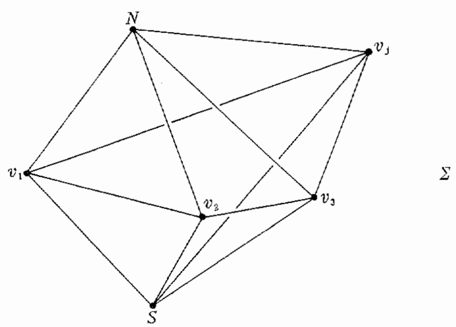

Fig. 1

Let $x = |N - S|^2$. All the equations to follow can be regarded as functions of the "variable" x. Let $A = N - S$, the axis, and let θ_j be the angle between the orthogonal projection of $v_j - S$ and $v_{j+1} - S$ into $A^\perp$. Let $e_j = v_j - S$, $e'_j = N - v_j$, $e''_j = v_{j+1} - v_j$, $j = 1, \cdots, n$, and call

$$e_{jj+1} = e_j \cdot e_{j+1} = \tfrac{1}{2}(e_j^2 + e_{j+1}^2 - e_j''^2) .$$

The e_j^2's and e_{jj+1}'s are intrinsic constant parameters describing Σ, functions of the lengths of the edges.

The structural equation comes from the fact that $\sum_{j=1}^{n} \theta_j$ is an integer multiple of 2π. This means that the suspension closes up as one proceeds around the equator. We use (1.1) to describe this fact in terms of the parameter x. With this in mind we define Q_j to be the real part of the numerator of (1.1), y_j the rest of the numerator, and H_jH_{j+1} as the denominator. Explicitly

$$\begin{aligned} Q_j &= e_{jj+1}x - (e_j \cdot A)(e_{j+1} \cdot A) \\ &= e_{jj+1}x - \tfrac{1}{4}(e_j^2 + x - e_j'^2)(e_{j+1}^2 + x - e_{j+1}'^2) , \\ y_j &= |A|\,[e_j, e_{j+1}, A]i , \\ H_j &= |A \times e_j| . \end{aligned} \tag{2.1}$$

Note

$$y_j^2 = -x \det \begin{pmatrix} e_j^2 & e_{jj+1} & \tfrac{1}{2}(e_j^2 + x - e_j'^2) \\ e_{jj+1} & e_{j+1}^2 & \tfrac{1}{2}(e_{j+1}^2 + x - e_j'^2) \\ \tfrac{1}{2}(e_j^2 + x - e_j'^2) & \tfrac{1}{2}(e_{j+1}^2 + x - e_j'^2) & x \end{pmatrix} ,$$

$$H_j^2 = e_j^2 x - \tfrac{1}{4}(e_j^2 + x - e_j'^2)^2 .$$

Geometrically

$$\sin \theta_j = \frac{1}{i} \frac{y_j}{H_jH_{j+1}} , \qquad \cos \theta_j = \frac{Q_j}{H_jH_{j+1}} .$$

This shows that

$$\begin{aligned} Q_j &= \text{a quadratic function of } x , \\ y_j^2 &= \text{a cubic function of } x , \\ H_j^2 &= \text{a quadratic function of } x . \end{aligned}$$

Thus by (1.1) we have

$$e^{i\theta_j} = \frac{Q_j + y_j}{H_jH_{j+1}} ,$$

$$1 = e^{i\Sigma\theta_j} = \prod_{j=1}^{n} \frac{(Q_j + y_j)}{H_jH_{j+1}} . \tag{2.2}$$

We also can write (2.2) as

$$\prod_{j=1}^{n} (Q_j + y_j) = \prod_{j=1}^{n} (Q_j - y_j) . \tag{2.3}$$

We refer to (2.3) as the basic structural equation of Σ.

3. The geometric properties of the extrinsic parameters, and generalized volume

3.1. Determination of y_j. Note that y_j is a function of the vectors e_j, e_{j+1}, e'_j, e'_{j+1}, e''_j and A, and no other part of the suspension. We claim

Property 1. *$y_j^2 = \frac{1}{4}e_j''^2 x(x - b_j)(x - b'_j)$, where b_j and b'_j are the real maximum and minimum values of x such that the two triangles $\langle N, v_j, v_{j+1}\rangle$ and $\langle S, v_j, v_{j+1}\rangle$ can be embedded in $\boldsymbol{R}^3$ with $x = |N - S|^2$.*

Proof. From the definition, y_j is only 0 when x is 0, or the two triangles above are planar, and this must occur when $x = b_j$ or b'_j. By the definition of y_j and (1.2) we see $y_j^2 = x \cdot$quadratic. Thus the roots of the quadratic are b_j and b'_j.

From (2.1) we see that the coefficient of x^3 is $\frac{1}{4}e_j''^2$. Note

$$e_j''^2 = \det \begin{pmatrix} e_j^2 & e_{jj+1} & 1 \\ e_{jj+1} & e_{j+1}^2 & 1 \\ 1 & 1 & 0 \end{pmatrix} .$$

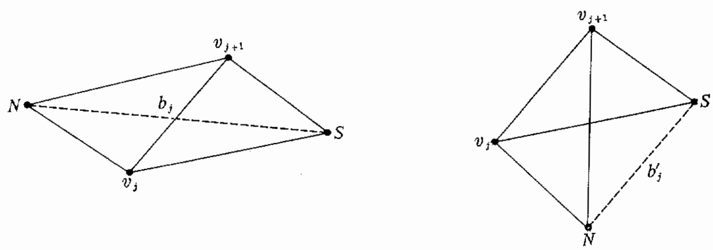

Fig. 2

3.2. Determination of Q_j and H_j.

Property 2. *Q_j and H_j^2 are quadratic functions of x with leading coefficient $-\frac{1}{4}$, and the other coefficients are polynomials in the squares of the edge lengths. (The main point is that they have the same leading coefficient).*

3.3. Topological and geometric properties. Let $w(\Sigma) = w$ denote the winding number of the equator about the N, S axis as mentioned in the introduction. Then from (2.2) we have

Property 3.

$$w = \sum_{j=1}^{n} \theta_j = \sum_{j=1}^{n} \frac{1}{2\pi i} \log \frac{Q_j + y_j}{H_j H_{j+1}} ,$$

where the θ_j and log *are chosen such that $-\pi < \theta_j \leq \pi$.*

Next we observe that the scalar triple product can be regarded as $\frac{1}{6}$ of the volume of the tetrahedron spaned by the three vectors. Thus,

Property 4. *The volume of* $\langle S, v_j, v_{j+1}, N\rangle$ *is*

$$\frac{y_j}{6i\sqrt{x}} .$$

The volume is considered as a signed quantity.

3.4. Generalized volume. Associated to any simplical oriented surface linearly mapped into $\boldsymbol{R}^3$ is a number which will turn out to be the volume enclosed by the surface in case it is embedded.

Let $v_1, v_2, \cdots$ be the vertices of the surface M in $\boldsymbol{R}^3$, and let $\langle v_j, v_k, v_l\rangle$ be a typical positively oriented 2-simplex of M. It is well-known that the volume of the tetrahedron spaned by 0, v_j, v_k, v_l is

$$\tfrac{1}{6}[v_j, v_k, v_l] = \tfrac{1}{6}\det(v_j, v_k, v_l) ,$$

where as before [, ,] is the scalar triple product, and implicit is that the volume may be $+$ or $-$. We define the volume "enclosed" by M as

$$V(M) = \tfrac{1}{6}\,\Sigma[v_j, v_k, v_l] ,$$

where the sum extends over all the 2-simplices of M, and the indices are chosen so that the orientation agrees with some orientation of M.

Lemma 1. *If M is embedded, then $\pm V(M)$ is the volume of region inside M.*

Proof. From the definition of $V(M)$ we see that each summand of $V(M)$ is just the volume of the cone over some 2-simplex σ of M with the sign chosen so that if the normal pointing away from the solid enclosed by the surface is on the opposite side of the plane determined by σ, then the sign is $+$ and is $-$ otherwise. This is compatible with some orientation of M.

Then it is easy to see that the volume enclosed by M is the sum of the volumes of the cones in some subdivision, where the signs are chosen $+$ or $-$ as above. Since $V(M)$ is invariant under subdivision the result follows.

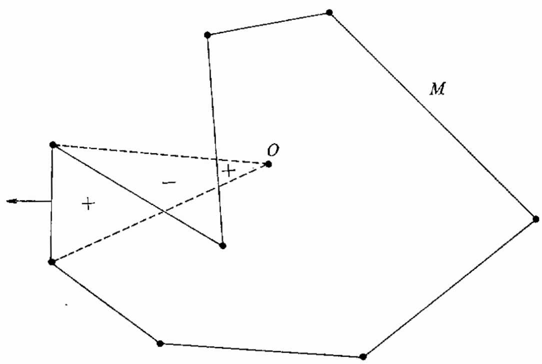

Fig. 3

Remarks. 1. $V(M)$ is independent of where the origin is chosen.

2. If M is immersed and bounds an immersed 3-manifold, then $V(M)$ is $\pm$ the volume of this 3-manifold. In particular $V(M) \neq 0$.

3. In the differentiable case the above formula is analogous to the Minkowski type formula

$$V(M) = \frac{1}{3}\int N \cdot X \, dA \, ,$$

where X describes the surface, and N is the outward pointing unit normal.

4. In case M is a suspension Σ, we can calculate $V(M)$ in terms of previously defined variables by taking the origin at S and using Property 4. For $x \neq 0$,

$$V(\Sigma) = \frac{1}{6i\sqrt{x}} \sum_{j=1}^{n} y_j \; .$$

4. Proofs of Theorems 1 and 2

4.1. The winding number theorem.

Proof of Theorem 1. From Properties 1 and 2 we see that θ_j is an analytic function of x, being the log of an algebraic function. The crucial observation is that θ_j is only a function of that part of the suspension which involves v_j and v_{j+1}, namely x, the five edge lengths, and the sign of y_j.

We next observe that if we consider x as complex and take the limit of θ_j as x goes to infinity along some path, in the complex plane, which avoids all the singular points of θ_j, we see that θ_j approaches some multiple w_j of 2π. This is because of Properties 1 and 2 that the highest degree of x (in a power series expansion about ∞ say) in the numerator and denominator is 2, and the leading coefficients are the same. Thus $e^{2\pi w_j i} = 1$, and w_j is some integer. It is clear that,

$$\sum_{j=1}^{n} w_j = w(\Sigma) \; .$$

But w_j depends continuously on the intrinsic parameters and thus can be computed for any particular case, and they all must be the same number. It is then clear that each $w_j \equiv 0$, and thus $w(\Sigma) = 0$.

4.2. Reducing the structural equation. Before we can show that $V(\Sigma) = 0$, we must investigate more thoroughly the nature of the structural equation (2.3). In particular we must show how it reduces or splits up into "disjoint" equations of the same sort. To this end we recall the b_j's and b'_j's of property 1 for y_j. We say y_{j1} is *equivalent* to y_{j2} (more precisely we should say j_1 is equivalent to j_2) iff $b_j = b_{j1}$ and $b'_{j2} = b'_{j1}$. Alternatively y_{j1} is equivalent to y_{j2} iff y_{j1}/y_{j2} is constant, as a function of x.

Lemma 2. *Let Σ flex with variable x, and let C be an equivalence class of indices described above. Then*

$$\prod_{j \in C} (Q_j + y_j) = \prod_{j \in C} (Q_j - y_j) . \tag{4.1}$$

Proof. Let b, b' correspond to the b_j's and b'_j's in the definition of C. Let C_0 be the set of indices corresponding to those y_j's which have $b_j = b$. We wish to show first that

$$\prod_{j \in C_0} (Q_j + y_j) = \prod_{j \in C_0} (Q_j - y_j) . \tag{4.2}$$

From Properties 1 and 2 we see that each factor of (2.3) is a nice algebraic function of x with no finite poles and only two branch points at b_j and b'_j. Start a path at any x not at any of the branch points, and proceed once around b and not around any of the other b_j's or b'_j's, (unless they are equal to b). Only the sign of the y_j's for $j \in C_0$ will change on both sides of (2.3). Thus

$$\prod_{N - C_0} (Q_j + y_j) \prod_{C_0} (Q_j - y_j) = \prod_{N - C_0} (Q_j - y_j) \prod_{C_0} (Q_j + y_j) , \tag{4.3}$$

where $N = \{1, 2, \cdots, n\}$.

Dividing (2.3) with (4.3) and cross-multiplying we get

$$\left[\prod_{C_0} (Q_j + y_j)\right]^2 = \left[\prod_{C_0} (Q_j - y_j)\right]^2 .$$

Thus (4.2) holds up to $\pm$.

If (4.2) holds with a minus sign, then we compute the coefficient of the highest power of x on the left and right (i.e., divide both sides by x^{2m}, m being the number of elements in C_0, and take the limit as $x \to \infty$). On the left it is $(-\frac{1}{4})^m$ and on the right it is $-(-\frac{1}{4})^m$, since the leading coefficient of Q_j is $-\frac{1}{4}$. Thus (4.2) must hold with a $+$.

Finally we repeat the above argument with (4.2) replacing (2.3), C replacing C_0, C_0 replacing N, and b' replacing b. (4.1) then follows.

4.3. The proof of Theorem 2. By Lemma 2 and Remark 4 after Lemma 1 it is sufficient to show that

$$\sum_{j \in C} y_j \equiv 0 .$$

Since all the y_j's in C differ by a multiplicative constant, write $y_j = c_j y = c_j x(x - b)(x - b')$. Thus we must show $\sum_{j \in C} c_j = 0$.

Expand (4.1) by the binomial theorem and collect the terms which do not cancel,

$$\left(2 \prod_C Q_j\right) \sum_C \frac{y_j}{Q_j} + \cdots = 0 , \tag{4.4}$$

where the terms left out involve higher powers of the y_j's and lower powers of the Q_j's. The order of x is 3/2 in y_j and is 2 in Q_j. Thus we see that the coefficient of the highest power of $x^{1/2}$ in (4.4) is

$$(-\tfrac{1}{4})^{m-1} \sum c_j \,.$$

Thus $\sum c_j = 0$ and applying the above argument for each equivalence class shows that $V(\Sigma) = 0$ proving Theorem 2.

4.4. The main result.

Corollary. *If Σ is a suspension which is immersed bounding an immersed 3 manifold, then Σ is rigid.*

Proof. If Σ is not rigid, it flexes by the definition of rigidity. If x varies during the flex, then $V(\Sigma) \neq 0$ by Remark 2, but this contradicts Theorem 2.

If on the other hand x is constant, we may regard Σ as being the union of n rigid tetrahedra $\langle N, v_j, v_{j+1}, S\rangle$ since all the edges are fixed during this flex. (Recall $x = |N - S|$.) The only way Σ can flex without extending to a congruence of $\boldsymbol{R}^3$ is for at least 2 of the v_j's to be situated on the north-south axis. But then Σ is immersed at neither N nor S. Thus in either case we obtain a contradiction, and Σ must be rigid.

5. Conjectures and related results

5.1. Conjectures. It is natural to conjecture that all immersed surfaces are rigid, but this is false (see [5]). The strongest conjecture along these lines which we can imagine is the following.

Conjecture 1. *If M^2 is a immersed 2-manifold in $\boldsymbol{R}^3$ bounding an immersed 3-manifold, then M is rigid.*

In view of Theorem 2 the following also seems natural.

Conjecture 2. *If M^2 is a simplical orientable 2-manifold linearly mapped into $\boldsymbol{R}^3$, then $V(M^2)$ is constant during any flex.*

5.2. Classifying flexible suspensions. The basic structural equation and particularly Lemma 2 can be used to give a geometric or algebraic geometric description of flexible suspensions. Property 1 can be regarded as defining y_j so that x and the y_j are elliptic functions of a single parameter. The group action defined on the elliptic curves defined by property 1 can then be used to generate many examples of flexible suspensions.

Each factor of (4.1) gives rise to four points on the elliptic curve, the roots of $Q_j + y_j$. The equation is to be interpreted as saying that the collection all the points is symmetric about the x-axis. It then becomes a combinatorial problem to find the ways of chosing such a collection of points. A flow graph with a flow in the elliptic group is helpful here.

5.3. The octahedron. The simplest nontrivial suspension is when $n = 4$, the octahedron, which can be viewed as a suspension in three ways. The analysis described in § 5.2 can be carried out in this case to give a complete description of all flexible octahedra. It turns out that there are three classes. The two simpler classes are described in [5].

However, in 1896 Bricard [2] also classified all flexible octahedra by a somewhat different approach. We thank Brank Grunbaum for pointing this reference and related ones out to us [1], [3], [7], [8].

We also wish to point out that Theorem 1 cannot be used to prove the corollary, even in the case of the octahedron. The following is a picture of an embedded (therefore rigid) octahedron with the property that if it is regarded as a suspension in any of the three ways, then the winding number of the equator about the north-south axis is zero.

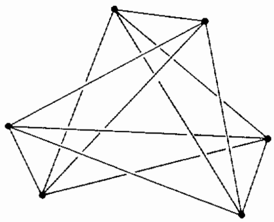

Fig. 4

5.4. More general suspensions. It is possible to generalize the above analysis of suspensions to the case where the equatorial curve is piecewise-smooth instead of piecewise-linear. One obtains by a similar analysis a formula analogous to (2.2) where an integral replaces the product. However, one must be a bit careful about what one means by rigidity here, for it is possible to have the suspension of an arc rigid if one does not allow a subdivision at a critical point on the arc. If one does allow subdivisions of this sort, the situation is quite similar to the piecewise-linear case. In fact it is possible to prove a winding number theorem with proof and statement similar to Theorem 1. We do not know how to show the analogue of Theorem 2.

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Added in proof. Conjecture 1 and the rigidity conjecture mentioned in the introduction are now known to be false; see [9] or [10].

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